SAMPLE SELECTION PROBABILITIES AND OPTIMAL SOFT MORPHOLOGICAL FILTERING

Pertti Koivisto[1] *and Pauli Kuosmanen*

Signal Processing Laboratory, Tampere University of Technology, Tampere, Finland

ABSTRACT

A new method of controlling the trade-off between noise attenuation and detail preservation in nonlinear filter design is presented. The technique is based on an appropriate combination of the sample selection probabilities of a filter with traditional error criteria. The practical applicability of the approach is empirically studied in connection with the training-based optimization of soft morphological filters. Also, the formulas for the sample selection probabilities of the basic soft morphological filters are derived.

1. INTRODUCTION

The trade-off between detail preservation and noise attenuation is one of the key issues in nonlinear filter design. The more emphasis is laid on the preservation of the details, the more noise tends to stay after filtering. In [1] and [3] the trade-off was controlled using the concepts of the breakdown probability and the breakdown point of the filter. There, the aim was not to force the filters to maximal noise removal but rather to preserve details while a desired amount of the noise was to be removed. This paper addresses filter design problems where the aim is the converse, noise attenuation under the requirement that a predetermined level of details is preserved. The detail preservation level is controlled by using the concept of the sample selection probability. As the experimental case we have used soft morphological filters.

2. SOFT MORPHOLOGICAL FILTERS

Soft morphological filters are stack filters with many desirable properties, e.g., they can be designed to preserve details well [4, 5]. The two basic soft morphological operations are *soft erosion* and *soft dilation*. Based on these operations, two compound operations, *soft opening* and *soft closing*, can be defined in the usual way.

Definition 1 The *structuring system* $[B, A, r]$ consists of three parameters, finite sets A and $B, A \subseteq B \neq \emptyset$, in $\mathbf{Z}^m$ (where $m \in \mathbf{Z}_+$ denotes the dimensionality of the signal space), and an integer r satisfying $1 \leq r \leq \max\{1, |B \backslash A|\}$. The set B is called the *structuring set*, A its (hard) *center*, $B \setminus A$ its (soft) *boundary*, and r the *order index* of its center or the *repetition parameter*.

The *translated set* T_x is defined by $T_x = \{x + t : t \in T\}$. A *multiset* is a collection of objects, where the repetition of objects is allowed. For example, $\{1, 1, 1, 2, 3, 3\} = \{3\diamondsuit 1, 2, 2\diamondsuit 3\}$ is a multiset. Soft morphological operations transform a signal $\mathcal{S}(\cdot)$ to another signal by the following rules.

Definition 2 Soft erosion (soft dilation) *of* $\mathcal{S}(\cdot)$ *by the structuring system* $[B, A, r]$ *is denoted by* $\mathcal{S} \ominus [B, A, r]$ $(\mathcal{S} \oplus [B, A, r])$, *where* $\mathcal{S} \ominus [B, A, r](x)$ $(\mathcal{S} \oplus [B, A, r](x))$ *is the* rth *smallest (largest) value of the multiset* $\{r\diamondsuit\mathcal{S}(a) : a \in A_x\} \cup \{\mathcal{S}(b) : b \in (B \setminus A)_x\}$.

Although there are no analytical criteria for deciding which soft operation (and with which parameters) is the best for some situation, a suitable operation and its parameters can be found using supervised learning methods, e.g., simulated annealing and genetic algorithms [2]. Of course, some training set, for which the desired output is known, is needed.

3. SAMPLE SELECTION PROBABILITIES

Let $(X_1, X_2, \ldots, X_N)$ denote a real-valued sample vector in the input window of a soft morphological filter. The definition of the soft morphological filters implies that the output is one of the input window samples. Thus, it is quite natural to ask what is the probability for the jth sample of the window to be the output. More exactly, the jth *sample selection probability*, $1 \leq j \leq N$, is denoted by $P[Y = X_j]$ and is defined as the probability that the output $Y = X_j$ [6].

In the case of an i.i.d. input signal it is quite simple to derive exact formulas for the calculation of the jth sample selection probability of the soft morphological erosion and dilation. To simplify the notations the following definition is given.

[1] On leave from the Department of Mathematical Sciences, University of Tampere.

Definition 3 *Let F be a soft erosion or soft dilation with the structuring system $[B, A, r]$. Then the* i.i.d. sample selection probability *of the set $D \subseteq B$ is denoted by $\varphi(F; D)$ and is defined by*

$$\varphi(F; D) = P[F(x) \in \{f(d) : d \in D_x\}],$$

where $f: \mathbf{Z}^m \to \mathbf{R}$ is some i.i.d. signal and x is any point in $\mathbf{Z}^m$.

If D has exactly one element Definition 3 reduces to that of the sample selection probability of an individual sample in the i.i.d. case. In this case the braces around the element can also be omitted, that is,

$$\varphi(F; \{b\}) = \varphi(F; b).$$

The next proposition gives the probability that in the i.i.d. case the output of a soft erosion or a soft dilation results from the samples corresponding to the soft boundary of the structuring set.

Proposition 1 *Let F be a soft erosion or soft dilation with the structuring system $[B, A, r]$. Then*

$$\varphi(F; B \setminus A) = \binom{|B \setminus A|}{r} \Big/ \binom{|B|}{r}.$$

Proof. Let F be the soft erosion with the structuring system $[B, A, r]$. Moreover, let $f: \mathbf{Z}^m \to \mathbf{R}$ be some signal satisfying the i.i.d. assumption and x be some point in $\mathbf{Z}^m$.

Let us denote

$$f_B = \{f(b) : b \in B_x\}$$

and

$$f_{B \setminus A} = \{f(b) : b \in (B \setminus A)_x\}.$$

By the definition of the soft erosion, $F(x) \in f_{B \setminus A}$ if and only if all of the r smallest elements of the set f_B belong to the set $f_{B \setminus A}$.

Now let us consider the distribution of the indices (in $\mathbf{Z}^m$) corresponding to the r smallest elements of the set f_B. Because $|B_x| = |B|$ there are

$$\binom{|B|}{r}$$

possible ways to place those indices in the set B_x. On the other hand, because $|(B \setminus A)_x| = |B \setminus A|$,

$$\binom{|B \setminus A|}{r}$$

of these events are such that all indices corresponding to the r smallest elements are located in the set $(B \setminus A)_x$. Hence, due to the i.i.d. assumption

$$\begin{aligned} \varphi(F; B \setminus A) &= P[F(x) \in f_{B \setminus A}] \\ &= \binom{|B \setminus A|}{r} \Big/ \binom{|B|}{r}. \end{aligned}$$

The proof for the soft dilation is similar, except that instead of the r smallest values we consider the r largest values. □

Because by the definition of soft morphological filters the output of a soft morphological filter is one of the input window samples, it holds that

$$\varphi(F; A) + \varphi(F; B \setminus A) = 1$$

if F is a soft erosion or soft dilation with the structuring system $[B, A, r]$. Thus, we now also have a formula for the probability that in the i.i.d. case the output a soft erosion or a soft dilation results from the samples corresponding to the hard center of the structuring set. However, the next proposition gives an alternative way to calculate the same probability.

Proposition 2 *Let F be a soft erosion or soft dilation with the structuring system $[B, A, r]$. Then*

$$\varphi(F; A) = \sum_{k=0}^{r-1} \frac{|A|}{|B| - k} \cdot \frac{\binom{|B \setminus A|}{k}}{\binom{|B|}{k}}.$$

Proof. Let F be the soft erosion with the structuring system $[B, A, r]$. Moreover, let $f: \mathbf{Z}^m \to \mathbf{R}$ be some signal satisfying the i.i.d. assumption, x be some point in $\mathbf{Z}^m$, and let us again denote

$$f_B = \{f(b) : b \in B_x\},$$

$$f_A = \{f(b) : b \in A_x\},$$

and

$$f_{B \setminus A} = \{f(b) : b \in (B \setminus A)_x\}.$$

Suppose now that $k \in \mathbf{Z}$ is a fixed value such that $0 \le k \le r - 1$ and let us consider the case in which none of the k smallest elements of the set f_B is in the set f_A. By the proof of Proposition 1 the probability of this case is

$$\binom{|B \setminus A|}{k} \Big/ \binom{|B|}{k}$$

(i.e., all of the k smallest elements of the set f_B are in the set $f_{B \setminus A}$). But now in the set f_B there are $|B| - k$ elements left, from which $|A|$ are contained in the set f_A. Thus, due to the i.i.d. assumption, the probability that the $(k+1)$th smallest element of f_B belongs to f_A is

$$\frac{|A|}{|B| - k}.$$

Hence, by the formula of the conditional probability the probability that the $(k+1)$th smallest element belongs to f_A with the condition that none of the smaller elements of f_B belongs to f_A is

$$\frac{|A|}{|B| - k} \cdot \frac{\binom{|B \setminus A|}{k}}{\binom{|B|}{k}}.$$

On the other hand, by the definition of the soft erosion, $F(x) \in f_A$ if and only if at least one of the r smallest elements of the set f_B belong to the set f_A. Thus,

$$\varphi(F; A) = \sum_{k=0}^{r-1} \frac{|A|}{(|B|-k)} \cdot \frac{\binom{|B \setminus A|}{k}}{\binom{|B|}{k}}.$$

The proof for the soft dilation is similar. □

Assuming an i.i.d. input signal, each sample in the soft boundary of the structuring set of a soft erosion or dilation has an equal probability to be the output, as has also each sample in the hard center of the structuring set of a soft erosion or dilation. Thus, Propositions 1 and 2 give immediately a way to calculate the selection probabilities of the samples of a soft erosion and dilation in the i.i.d. case. The next corollary summarizes these results.

Corollary 1 *Let F be a soft erosion or soft dilation with the structuring system $[B, A, r]$. Then*

$$\varphi(F; b) = \begin{cases} \sum_{k=0}^{r-1} \frac{1}{(|B|-k)} \cdot \frac{\binom{|B \setminus A|}{k}}{\binom{|B|}{k}}, & \text{if } b \in A; \\ \frac{\binom{|B \setminus A|}{r}}{|B \setminus A| \binom{|B|}{r}}, & \text{if } b \in B \setminus A. \end{cases}$$

For soft openings and closings the situation is more complex even in the i.i.d. case, because the intermediate signal is not necessarily an i.i.d. signal. Then, the formulas for the sample selection probabilities of the soft opening and the soft closing derived directly using only the above formulas are not exact, but they may still yield a good estimate.

4. OPTIMIZATION UTILIZING SAMPLE SELECTION PROBABILITIES

When sample selection probabilities are used to control the detail preservation ability of a filter, the most important selection probability is the *midpoint selection probability*, i.e., the selection probability of the sample corresponding to the origin of the input window. The larger the midpoint selection probability is, the less the input signal alters in the filtering. Thus, in order to have a good detail preservation ability, a soft morphological filter must have a relatively high midpoint selection probability.

In our approach the midpoint selection probabilities are used as constraints when optimal soft morphological filters are sought for some specific situation. That is, the midpoint selection probability of the optimal filter should be over a predetermined level.

The actual optimization is done using a test signal pair where the noisy signal is the source signal and the desired signal is the target signal. Thus, in order to have a low error, the optimal filter must remove noise with efficiency. The noise removing ability of the optimal filter is controlled by the error criterion used in the optimization. However, a criterion that concentrates only on the noise removal, e.g. the breakdown probability, usually does not produce unique solutions. Thus, a good error criterion for the optimization is now such that it pays most attention to noise removal without totally ignoring detail preservation.

The feasibility of any filter with respect to the midpoint selection probability condition is measured simply by calculating whether the filter satisfies this condition. If it does not, a penalty is added to the error of the actual test image.

For soft erosions and soft dilations it is usually possible to use the formulas derived above. Thus, the midpoint selection probability condition for a soft erosion and a soft dilation can be expressed as a function

$$B(F; \beta) = \begin{cases} 0, & \text{if } \varphi(F; \mathbf{0}) \geq \beta; \\ 1, & \text{otherwise;} \end{cases}$$

where β is the desired midpoint selection probability level.

If it is not possible (or reasonable) to use the formula in a closed form to calculate the midpoint selection probability, a test signal pair is needed for the calculation of an estimate of the midpoint selection probability. This estimate is then used to justify whether the filter satisfies the given constraint.

The details of the use of the midpoint selection probability condition as the constraint in the optimization are the same as with the breakdown point and breakdown probability constraints. Due to the limited space the details are omitted (cf. [1, 3]).

5. EXPERIMENTAL RESULTS

The experimental tests in this paper are based on the following 2-D test case. The training image is a 505×20 part of the 512×512 gray-level image "Harbour", with $(4, 240)$ as the coordinates of the upper left corner. The noisy training image is corrupted by positive impulsive noise with probability 0.1. Here, a positive impulse means the largest possible pixel value. The optimized filters are soft erosions with structuring sets included in the 5×5 square.

In the test case optimal soft erosions with different midpoint selection probability levels were found. As the error criterion we used the MSE, which was now a simple choice because the optimal soft erosion under the MSE (without any constraints) removed practically all impulses. The midpoint selection probability of the optimal soft erosion found was $5/9$. Thus, in this experiment reasonable midpoint selection constraints are larger than $5/9$.

Figure 1 shows the error between the original training image and the filtered noisy training image when the desired midpoint selection probability level of the optimal soft erosion varies from 0.55 to 1.0 (the largest possible). The soft

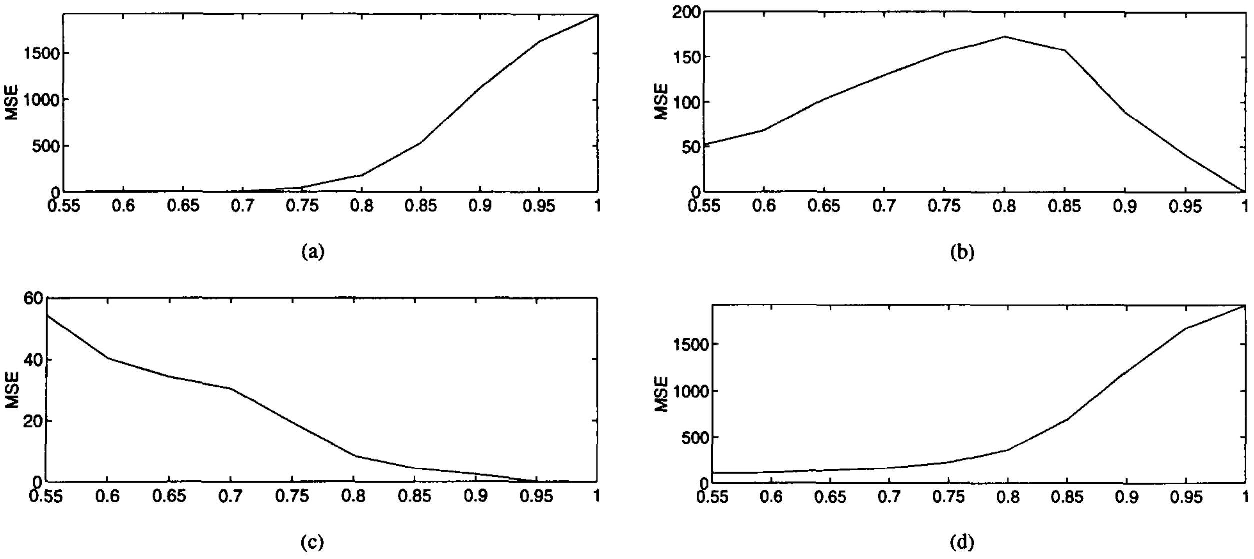

Figure 1: The MSEs between the original training image and the filtered noisy training image as a function of the midpoint selection probability level. (a) The MSE caused by the impulses not removed, (b) The MSE caused by the removed impulses, (c) The MSE caused by the filtering of the noncorrupted pixels, (d) The total MSE.

erosion with midpoint selection probability 1.0 is the identity filter. Thus, the rightmost value in Figure 1d is the original MSE between the noisy and noise-free training image. As mentioned above, the midpoint selection probability of the optimal soft erosion found under the MSE (without any constraints) was larger than 0.55. Therefore, the total error (Figure 1d) is smallest when the desired level is 0.55. When the desired midpoint selection probability level increases, the total MSE increases, first slowly and then more and more rapidly. The reason for this is that with moderate midpoint selection probability levels, the optimal filter is still capable of removing almost all of the impulses, but when the desired level increases, more and more impulses remain in the image after the filtering.

Figures 1a-c give a more thorough insight on the error components. As can be seen, the dominant part of the total error is the one that is caused by the impulses that are not removed. The other two components have a significant role only if the error caused by the remaining impulses is small. Moreover, the curve of the error caused by remaining impulses (Figure 1a) has the same shape as the curve of the total MSE.

It is worth noticing that when the desired midpoint selection probability level is less than or equal to 0.7 there is no difference in the error caused by remaining impulses. In fact, in each of these cases almost all of the impulses are removed. The increasing trend in the total MSE is now explained by the increasing error after impulse removal in the impulse locations (Figure 1b). The reason for the increased error after impulse removal is that (almost) perfect impulse removal together with larger midpoint selection probability levels leads to the use of larger structuring sets, which, on the other hand, results in the increased average error when the impulse is replaced with a non-corrupted value. Thus, if the number of the impulses removed does not decrease or decreases only slightly the total MSE caused by the removed impulses increases. With larger midpoint selection probability levels than 0.8 the number of removed impulses decreases rapidly. Thus, also the error caused by the removed impulses starts to decrease.

The detail preservation ability of the optimal soft erosion can be seen from Figure 1c, which shows the error caused by the filtering of the noncorrupted pixels. As can be seen, the curve is now the opposite when compared to that of Figure 1a. That is, when the desired midpoint selection probability level increases, also the detail preservation ability of the filter increases.

Figure 2 shows in qualitative and visual terms the results of our experiment. From left to right Figure 2 illustrates the effect of requiring that the filters have an increasing midpoint selection probability. The image is divided into 10 regions filtered with optimal filters with the midpoint selection probability constraint being $0.55, 0.60, 0.65, \ldots, 1.0$. It can clearly be seen that the filters with a smaller midpoint selection probability remove disturbing noise more efficiently than the others, but their drawback is the heavier blurring visible in Figure 3.

Figure 3 represents the difference between the original noise-free test image and its filtered counterpart, using the same filters as in Figure 2. The image obviously shows that

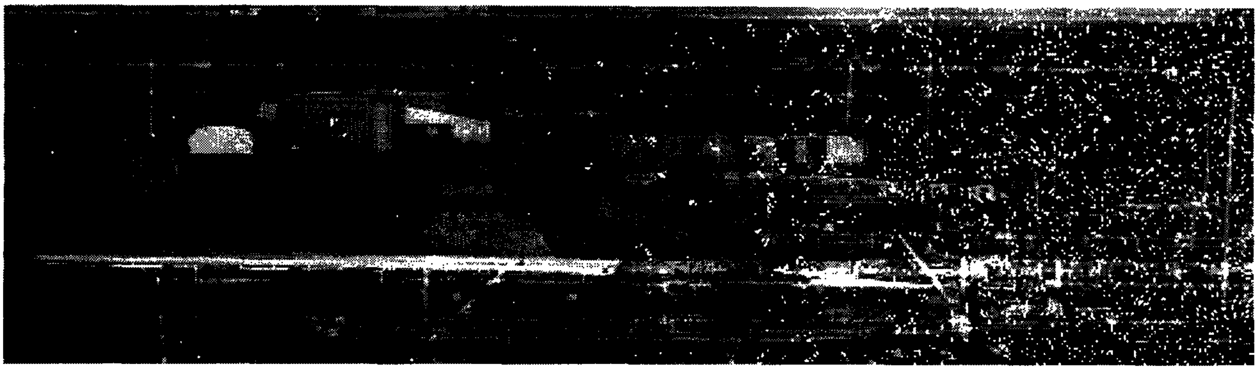

Figure 2: The noisy training image filtered by optimal filters with midpoint selection probability levels $0.55, 0.60, 0.65, \ldots, 1.00$.

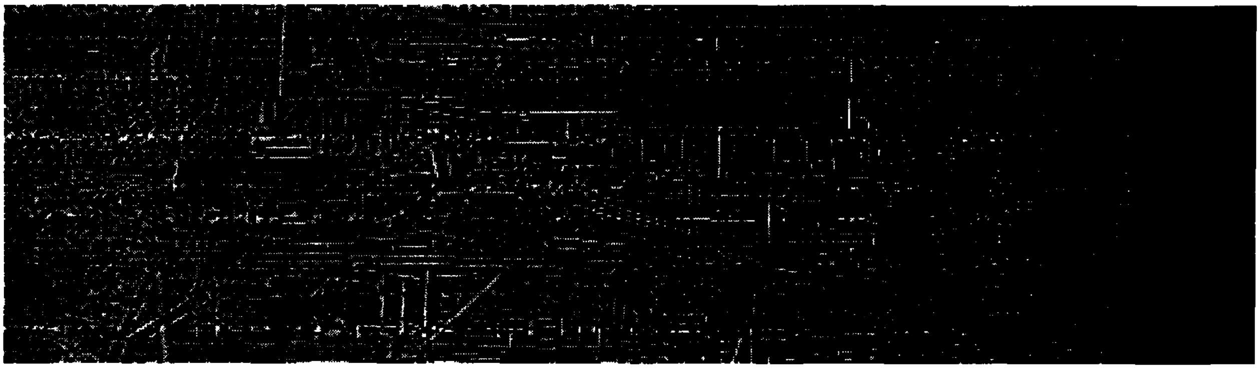

Figure 3: The difference between the original image and its filtered counterpart by optimal filters with midpoint selection probability levels $0.55, 0.60, 0.65, \ldots, 1.00$. For visualization purposes, the image has been gamma corrected before printing with gamma value $\gamma = 10$.

many more small details have been removed with filters with a smaller midpoint selection probability (the leftmost end) than with filters with large midpoint selection probabilities (the rightmost end). Even the shapes in the image can be detected in the left hand side of the image. Figures 2 and 3 may be summarized: a large midpoint selection probability level guarantees high efficiency in terms of detail preservation with a drawback of remaining impulses; to alleviate this a moderate midpoint selection probability level can provide a good compromise.

6. REFERENCES

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