NON-LINEAR INTERPOLATION OF MISSING IMAGE DATA USING MIN-MAX FUNCTIONS

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ABSTRACT

This paper proposes the use of min-max functions as a model for the interpolation and reconstruction of missing image data. It is shown how an interpolation equation based on these functions is formed and differentiated. The resulting solution is not closed form, therefore the derived gradient expressions are employed as part of various numerical optimisation schemes. Different interpolations can be found by using the squared and absolute errors - the latter presents a more complicated solution which is discussed. Results are shown for the interpolation of missing data in an image. These are compared with interpolants derived using a 2D AR model and conclusions are drawn about the suitability of the new technique for reconstructing different image features.

1. INTRODUCTION

Bright and dark blotches that represent areas of missing pixels are types of degradation that appear frequently on film and video. Often the damage is bad enough that the original image data is completely obliterated. This paper proposes an interpolation scheme based on min-max functions for reconstructing such areas of missing data.

Previous work has involved interpolating the missing data using an AR model/predictor as a basis for the interpolation [3],[1]. This method reconstructs fine image detail well, but unfortunately smooths out edges, which are very important for the visual appreciation of an image. The aim of the work is to demonstrate the interpolation capabilities of min-max based interpolation and to show that the scheme performs differently to the AR based method, principally with better reconstruction of edges.

The use of the min-max predictor produces a system of non-linear equations which are solved numerically. The solution is made more efficient by employing the analytic gradient expressions that this paper shows how to calculate. These expressions also show that local minima can arise, which complicates the optimisation strategy. The strategy used to produce the results in this paper is described in section 4.

The presentation of results in this paper is split into two parts. The first discusses interpolation using the squared error and presents a reconstruction of two images using min-max interpolation and AR interpolation.

The second section presents the use of the absolute error for min-max based interpolation. Results demonstrate how this differs from the use of the squared error.

Finally, conclusions are drawn in section 7 about the suitability of min-max based interpolation for the reconstruction of missing data in images.

2. INTERPOLATION

Linear interpolation using a multidimensional AR model/predictor is used in [3]. The underlying assumption in this interpolation scheme is that the interpolated data can be determined by choosing values that minimise the sum of the squared prediction errors over the missing area. The model coefficients are estimated using weighted least squares to annul the effect of the missing data. It is then possible to derive a closed form solution to the resulting set of interpolation equations. Detecting the missing areas can be posed as a separate problem, discussed in [2].

This paper proposes the use of the min-max function as the predictor in a non-linear interpolation scheme. An appropriate min-max function for a set of data can be determined using stack filtering methods ([6], [4] and [7]).

The min-max signal model - considering here a 1-D signal, y, for simplicity - is

$$y(n) = f_m(\ldots, y(n-2), y(n-1), y(n+1), \ldots) + \varepsilon_n \quad (1)$$

where f_m is a min-max function of pixels not including $y(n)$ and ε_n is some prediction error (or excitation or residual). A prediction equation, (1), may be written at every pixel site that involves a missing pixel either at that site, or in the support of the predictor. Then, the sum of the squares of the prediction errors, $E = \sum_n \varepsilon_n^2$, is minimised with respect to the values of the missing pixels.

It is possible to derive analytic expressions for the derivatives of E and this is described in next section.

3. DIFFERENTIATING THE INTERPOLATION EQUATION

The first step to differentiating E uses the chain rule:

$$\frac{\partial E}{\partial y(i)} = 2\sum_n [(y(n) - f_m(\ldots, y(n-2), \quad (2)$$

$$y(n-1), y(n+1), \ldots)) \times \frac{\partial(-f_m)}{\partial y(i)}]$$

where $y(i)$ is one of the missing pixel values.

The next step is to determine $\frac{\partial(-f_m)}{\partial y(i)}$. For this the following results are used:

$$\partial(max(y(i), k))/\partial y(i) = u(y(i) - k) \quad (3)$$

$$\partial(min(y(i), k))/\partial y(i) = u(k - y(i)) \quad (4)$$

where $u(k)$ is the unit step function and k is a min-max function that does not depend on $y(i)$.

The difficulty that remains is to write down the derivative of a given min-max function using these standard forms. A min-max function, f_m, can be written as the maximum of a series of minimums, some of which depend on $y(i)$ and some of which do not, for example

$$max(min(g_1), min(g_2), min(g_3, y(i)), min(g_4, y(i))) =$$

$$max(min(g_1), min(g2), min(max(min(g_3), min(g_4)), y(i)))$$

where g_i denotes a list of variables that does not include $y(i)$ (for example $y(n-3), y(n-1), y(n+1)$).

It can be shown that

$$\frac{\partial f_m}{\partial y(i)} = u(y(i) - max(min(g_1), min(g_2))) \times u(max(min(g_3), min(g_4)) - y(i)) \quad (5)$$

The proof of this result involves realising that the following conditions must apply for f_m to be $y(i)$ and the derivative equal to 1:

$$max(min(g_3), min(g_4)) > y(i)$$

and

$$max(min(g_1), min(g_2)) < y(i)$$

and that in all other cases f_m is not $y(i)$ and the derivative is 0.

The method outlined above can also be used to differentiate a min-max based interpolation equation formulated using the absolute error. The following formula will be useful for the first stage of this differentiation:

$$\frac{\partial \mid f_m \mid}{\partial y(i)} = sign(f_m)\frac{\partial f_m}{\partial y(i)} \quad (6)$$

The minimum of the error equation is to be found when the differential is zero. Examination of a typical differential shows that it is possible to have more than one minimum. Some occur when one or more of the unit step functions equal zero. By differentiating again these can often be shown to correspond to points of inflexion. However, the equations show that they can also correspond to local minima. This is an important consideration when choosing the optimisation strategy.

4. OPTIMISATION STRATEGY

Minimising an error equation based on a min-max function requires much care. Gradient information can help greatly to find the route to the minimum, but sometimes local minima occur, so the strategy chosen must reflect this.

The search for the minimum is normally performed using simulated annealing (described below). This technique is used to avoid converging to a local minimum. Conjugate gradient methods (described below) are used to find the minimum when it is thought that local minima will not be encountered. These methods use gradient information and are more efficient than simulated annealing. In the results presented next, only the reconstruction of the hair texture of the mandrill exhibited local minima and required simulated annealing. The smoother areas were interpolated using conjugate gradient methods - as a check they were compared to results using simulated annealing, which took longer to obtain.

Each method was implemented using the algorithms supplied with the book, Numerical Recipes in C [5]. When using conjugate gradient methods the derivatives must be used during the line minimisation and it is necessary to add a check on the derivative in order to ensure that the method does not stop at points of inflexion. If it is found that a derivative that is equal to zero corresponds to a point of inflexion, the derivative is set to the value of the derivative below this point.

The simulated annealing algorithm is based around an algorithm that transforms the points of a simplex so as to search for the route that minimises the error. In order to avoid local minima the points are randomly fluctuated each time they are observed and new points are accepted if they lead to an error that is less than a sum of the previous error and a positive random fluctuation. The amount of fluctuation is controlled by a parameter known, in analogy to physical processes, as the temperature.

At each temperature the number of iterations performed is equal to 50 times the number of missing pixels. The temperature is reduced by 20% at a time, from an initial value equal to twice the maximum error encountered for all the points in the initial simplex, until the temperature is less than 0.5. At this point the temperature is reduced to 0 and the minimisation continued using conjugate gradient methods.

These methods are very computationally expensive, especially when compared to the closed form solution of a system of linear AR based interpolation equations.

5. RESULTS: SQUARED ERROR

Figure 1 shows interpolation results for three edge features in an image called 'regions'. The areas interpolated are shown as bright squares in the first of these images.

Conjugate gradient methods were used to minimise the min-max interpolation equation and the predictor used was a seven by seven median predictor, with the top left-hand position missed out to give an odd number of positions in the support. The position of the predicted pixel was located at the centre of the support. Around 25 iterations were required in each case until all the gradients became zero

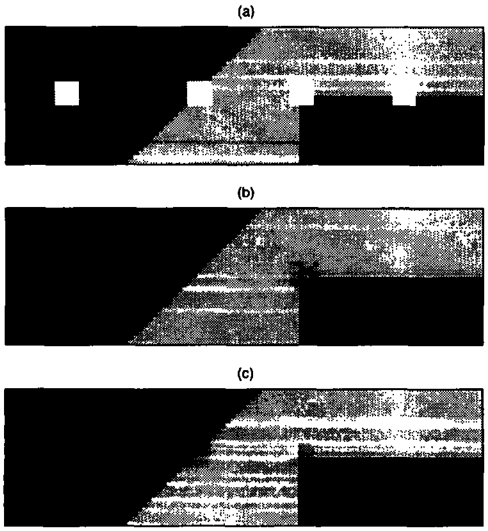

Figure 1: (a) Locations of missing data. (b) Detail of image reconstructed using min-max based interpolation. (c) Detail of image reconstructed using AR based interpolation.

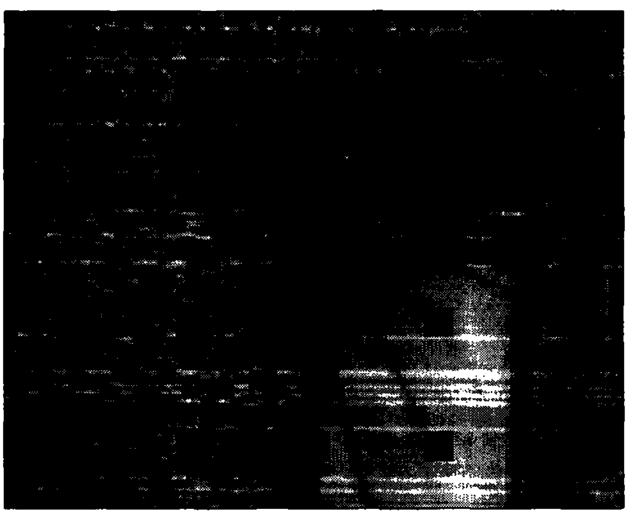

Figure 2: Locations of missing data. (Some blocks have been placed over edge features.)

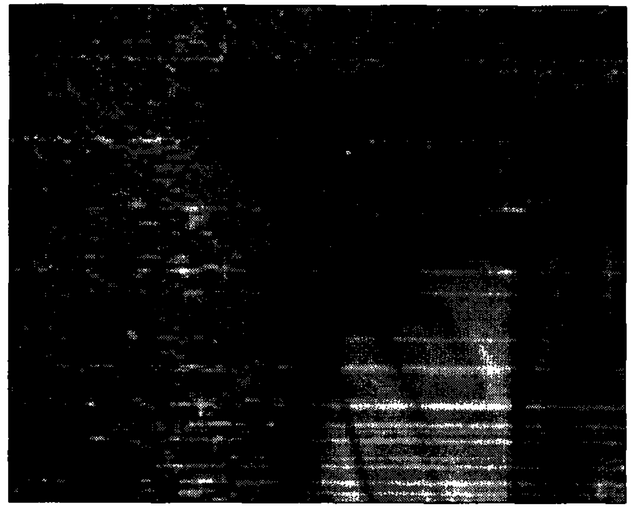

Figure 3: Detail of image reconstructed using min-max based interpolation.

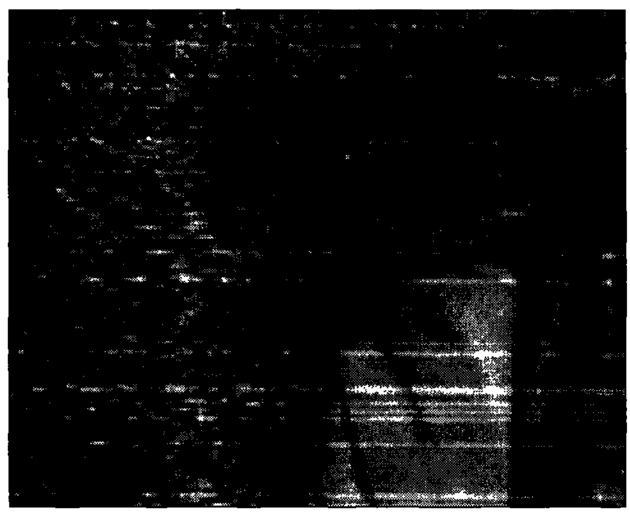

Figure 4: Detail of image reconstructed using AR based interpolation.

Figure 5: Absolute difference between image reconstructed using min-max based interpolation and the original image (scaled by a factor of 2).

Figure 6: Absolute difference between image reconstructed using AR based interpolation and the original image (scaled by a factor of 2).

or very small, except for the corner feature which required 40. The initial values were chosen randomly from a uniform distribution between 0 and 255.

In the case of the AR interpolation the same predictor support was used except that the top left-hand position was not missed out. Note that for the AR interpolation, which involves matrix inversion, a small amount of noise was added to the image to avoid ill-conditioned matrices.

These figures illustrate that the min-max based interpolator interpolates edge features very well, but both methods have problems with the corner feature. It also shows that the AR interpolation sometimes has problems with edges. Although the coefficients were chosen to ensure that the vertical and horizontal edges were well reconstructed, this was not possible for the diagonal edge and the result is a smudge.

Figure 2 shows a detail of the image 'mandrill' with areas of missing data, 6 by 6 pixels in size, and figure 3 shows the image reconstructed using min-max based interpolation. Simulated annealing was used to minimise the interpolation equation. The predictor used is a seven by seven median predictor, with the top left-hand position missed out to give an odd number of positions in the support. The position of the predicted pixel is located at the centre of the support.

Figure 4 shows the image reconstructed using AR based interpolation. The same predictor support was used (except that the top left-hand position was not missed out).

Figures 5 and 6 are difference images showing the absolute difference between the reconstructed images and the original images. The mean absolute errors over the reconstructed areas are 19.66 for the min-max based interpolation and 24.02 for the AR based interpolation.

In general the interpolated regions for both methods appear to fit in well with the rest of the image. It appears from the difference image that the min-max based interpolations fit in better - however the extra activity shown in the difference image of the AR interpolation in the blocks corresponding to the hair texture actually highlights the better reconstructions. It is perhaps surprising that an interpolation scheme based on a median predictor can reconstruct the hair texture so well, though at times it looks a little flatter than the AR result. The AR interpolation has reconstructed the hair texture well, but has not performed as well as the min-max interpolation in the block on the left of the eye and the block over the diagonal strip on the flat region to the left of the nose.

6. RESULTS: ABSOLUTE ERROR

Using the absolute error for min-max based interpolation does not necessarily produce one optimal interpolant. For example, using the predictor $y(n) = median(y(n-1), y(n-2), y(n-3))$ with the step sequence

$$0, 0, 0, *, 100, 100, 100$$

where $*$ indicates a missing pixel , the optimal interpolant is found to be any value between 0 and 100. However, note that appropriate values for the interpolant are the values at the extremes, namely 0 or 100.

With two missing pixels in the sequence

$$0, 0, 0, *, *, 100, 100, 100$$

the minimum of the error surface is a triangular plateau. The vertices of the triangle are $(0, 0)$, $(0, 100)$ and $(100, 100)$. All these values are appropriate interpolants of the missing pixels.

For more complex cases, such as with images, one global minima does often occur. In cases where it does not the sum squared gradient around the interpolated region is employed to determine a unique interpolant. The implementation involves starting an optimisation routine at the ambiguous result, but employing this different error criterion to explore the plateau. The search is not allowed to increase the absolute error.

Regions of constant error can also occur in the error space at locations that are not at the minimum of the function. Simulated annealing was used to obtain the results obtained in this section, but if conjugate gradient methods are used then this needs to be taken into account.

Figure 7 shows the 'regions' image from the previous section. Once again it is interpolated using the same 7 by 7 median predictor as in the previous section, but this time the interpolation is based on the absolute error. In every case except the corner, multiple solutions were found and the most appropriate was determined according to the method described above.

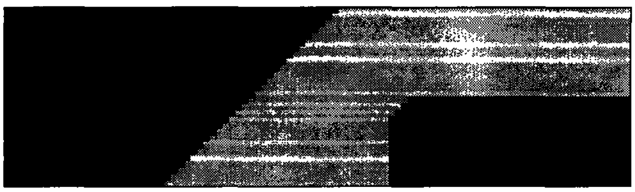

Figure 7: Regions image reconstructed using min-max based interpolation and the absolute error.

In the case of the corner the method has very neatly cut the corner off. This illustrates that the use of the absolute error can lead to interpolations that are less smudged.

7. CONCLUSION

This paper has shown how to construct a non-linear interpolator based on min-max functions and how to calculate the derivative of a given min-max based error equation. The derivatives show that the minimisation of the error equation is a complicated problem, but one which can be solved using a combination of simulated annealing and conjugate gradient methods.

The results presented for the squared error case show that the min-max based interpolator gives a pleasing reconstruction that performs well for different image features. Results for standard applications of min-max filters or recursive min-max prediction were not shown as it is clear that they cannot perform as well for this problem.

The results for min-max based interpolation were similar to those obtained using AR based interpolation, though this had the advantage of being able to optimise its coefficients. The main differences are that sometimes (though not always) the hair texture on the mandrill is worse when reconstructed with min-max based interpolation, but edge features are reconstructed better.

Min-max based interpolation using the absolute error criterion involves a few more considerations because there are regions of equal error in the error space. The paper has proposed solutions to the problems that this gives and shows that the use of the absolute error can lead to less smudging of reconstructed edge features.

The huge amount of computation is an important factor that limits the usefulness of min-max based interpolation, especially as AR based interpolation yields an efficient closed form solution.

(More information can be found along with examples of reconstructed images, on the main author's webpage:
http://www2.eng.cam.ac.uk/~sa210/index.html)

8. REFERENCES

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